Solutions to JEE Advanced Booster Test - 6 | 2024 | Code A

[PHYSICS]

SECTION 1

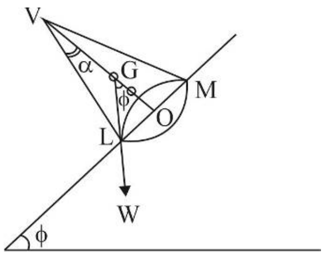
1.(D) Centre of gravity G of the cone will just fall within the base of the cone i.e., will pass through L

$$\tan\phi = \frac{LO}{OG} = \frac{r}{h/4}$$

Where, r is the radius of the base and h is the height of the cone.

Now, if the cone is on the point of slipping as well as turning over at the same time, then $\lambda = \phi$ or $\tan\lambda = \tan\phi$

or $\quad \frac{1}{\sqrt{3}} = 4\tan\alpha$



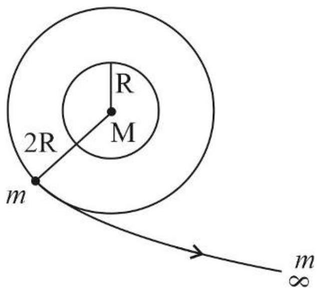
or $\quad \tan\alpha = \frac{1}{4\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{4\sqrt{3}}\right)$

$\therefore \quad$ Vertical angle of the cone $= 2\alpha = 2\tan^{-1}\left(\frac{1}{4\sqrt{3}}\right)$

2.(C) Total energy of particle should be made zero to escape to infinity.

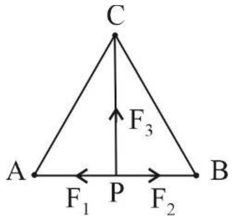
Total energy of particle inside satellite $= -\frac{GMm}{4R}$

$$-\frac{GMm}{4R} + \frac{1}{2}mV^2 = 0$$



$$\Rightarrow \quad V = \sqrt{\frac{GM}{2R}}$$

3.(C) When the particle is at point P, F_1 and F_2 are equal and opposite and hence will cancel each other. Therefore, the resultant force is only F_3, *i.e.*,



$$F = F_3 = \frac{Gm^2}{\left(\frac{\sqrt{3}}{2}l\right)^2} = \frac{4Gm^2}{3l^2}$$

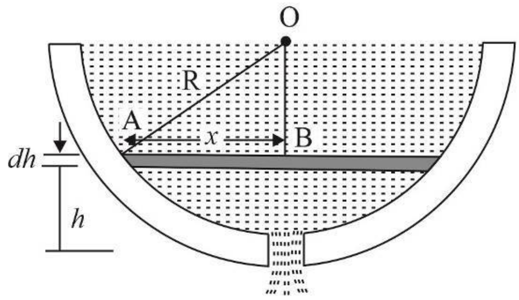
4.(B) Volume of liquid in layer of thickness *dh*,

$$dV = \pi x^2 dh$$

In ΔOAB

$$x = \sqrt{R^2 - (R-h)^2} = \sqrt{2Rh - h^2}$$

$$\therefore \quad dV = \pi\left(2Rh - h^2\right)dh$$



By equation of continuity, $a_1 v_1 = a_2 v_2$

$$\Rightarrow \quad -\frac{dV}{dt} = a\sqrt{2gh}$$

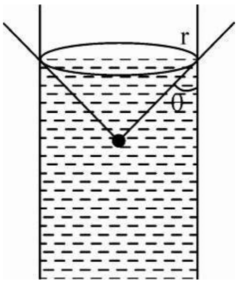
$$\Rightarrow \quad -\frac{\pi\left(2Rh - h^2\right)}{a\sqrt{2gh}}dh = dt$$

$$\Rightarrow \quad \int_0^T dt = \int_R^0 \frac{-\pi}{a\sqrt{2g}}\left(2Rh^{1/2} - h^{3/2}\right)dh$$

$$\Rightarrow \quad T = \frac{-\pi}{a\sqrt{2g}}\left[\frac{4}{3}Rh^{3/2} - \frac{2}{5}h^{5/2}\right]_R^0 \qquad \therefore \quad T = \frac{7\sqrt{2}\pi R^{5/2}}{15a\sqrt{g}}$$

5.(B)

$$\left.\begin{aligned} &2\pi r\cos(\cos^{-1}3/4) = (\pi r^2 \times 6)\,\rho g \quad .....(i)\\ &2\pi r\cos\theta = (\pi r^2 \times 4)\rho g \quad .....(ii)\end{aligned}\right] \text{Solving } (i) \text{ and } (ii) \text{ we get } \theta = 60^\circ$$



6.(AB) Loss in P.E. = Gain in KE.

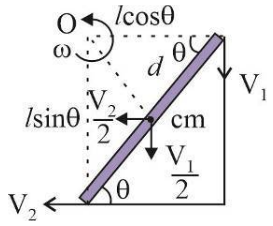
$$\Rightarrow\ mg\frac{l}{2}-mg\frac{l}{2}\sin 30^\circ=\frac{1}{2}mV_{cm}^2+\frac{1}{2}I_{cm}\omega^2$$

$$\Rightarrow\quad \frac{mgl}{4}=\frac{1}{2}mV_{cm}^2+\frac{1}{2}\frac{ml^2}{12}\omega^2$$

$$\Rightarrow\quad gl=2V_{cm}^2+\frac{\omega^2 l^2}{6}\qquad \ldots(1)$$

Let d = distance between inst. axis of rotation 'O' and c.m.

$$\frac{V_{cm}}{d}=\frac{\sqrt{V_1^2+V_2^2}}{2d}=\omega=\frac{V_1}{l\cos\theta}=\frac{V_2}{l\sin\theta}=\frac{V_{cm}}{d}$$



$$\Rightarrow\quad \frac{\sqrt{\omega^2 l^2\cos^2\theta+\omega^2 l^2\sin^2\theta}}{2d}=\frac{V_{cm}}{d}$$

$$\left[\because\ V_{cm}=\sqrt{\frac{V_1^3}{4}+\frac{V_2^2}{4}},V_{cm}=\sqrt{\frac{V_1^2+V_2^2}{4}}\right]$$

$$\Rightarrow\quad \frac{\omega l}{2d}=\frac{V_{cm}}{d}$$

$$\therefore\quad V_{cm}=\frac{\omega l}{2}\qquad \ldots(2)$$

By (1) and (2), $gl=\frac{\omega^2 l^2}{2}+\frac{\omega^2 l^2}{6}$

$$\Rightarrow\ gl=\frac{4\omega^2 l^2}{6}\qquad \therefore\ \omega=\sqrt{\frac{3g}{2l}}$$

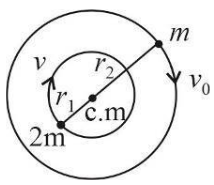
7.(AC) $\because\ m_1 r_1=m_2 r_2$

$\Rightarrow\ 2mr_1=mr_2$

$\therefore\ r_2=2r_1$

$\because\ r=r_1+r_2$

$\therefore\ r=3r_1$



(a) $\therefore\ r_1=\frac{r}{3}$ and $r_2=\frac{2r}{3}$

(b) Angular speed is same $\therefore\ \omega=\frac{v_0}{2r/3}=\frac{v}{r/3}$ $\qquad\therefore\ v=\frac{v_0}{2}$

$\frac{T_1}{T_2}=1:1$ as angular speeds are same.

8.(D) Let the volume and density of block be v & ρ_s and density of liquid be ρ_ℓ

On Earth : $w_1 = v\rho_s g$, $w_2 = v\rho_s g - v\rho_\ell g = v(\rho_s - \rho_\ell)g$, $w_1 - w_2 = v\rho_\ell g$, $w_1 / w_2 = \rho_s / (\rho_s - \rho_\ell)$

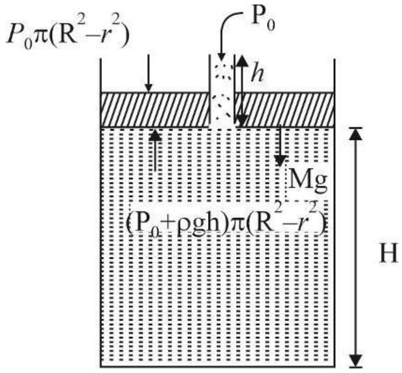
On moon : $w_1 = v\rho_s g / 6$, $w_2 = v\rho_s g / 6 - v\rho_\ell g / 6 = v(\rho_s - \rho_\ell) g / 6$,

$w_1 - w_2 = v\rho_\ell g / 6, w_1 / w_2 = \rho_s / (\rho_s - \rho_\ell)$

9.(BC) M = 3kg

R = 4cm

r = 1 cm



$$P_0\pi\left(R^2 - r^2\right) + Mg = \left(P_0 + \rho gh\right)\pi\left(R^2 - r^2\right)$$

$$\Rightarrow\ Mg = \rho gh\pi\left(R^2 - r^2\right)$$

$$\Rightarrow\ \rho gh = \frac{Mg}{\pi\left(R^2 - r^2\right)}$$

$$\Rightarrow\ h = \frac{M}{\rho\pi\left(R^2 - r^2\right)} = \frac{3}{10^3\pi(16-1)10^{-4}} m$$

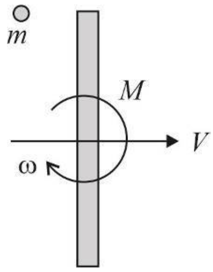
$$\therefore\ h = \frac{2}{\pi} m$$

For equilibrium of liquid, $mg + P_0\pi r^2 + \left(P_0 + \rho gh\right)\pi\left(R^2 - r^2\right) = \left\{P_0 + \rho g\left(H + h\right)\right\}\pi R^2$

$$\Rightarrow\ mg + \rho gh\pi\left(R^2 - r^2\right) = \rho g\left(H + h\right)\pi R^2 \ \Rightarrow\ m = \rho\pi\left(HR^2 + hr^2\right)$$

$$H = \left(\frac{m}{\rho\pi} - hr^2\right)\frac{1}{R^2} = \frac{11}{32\pi} m$$

10.(ABC) $P_i = P_f$



$$\Rightarrow\ mv_0 = MV$$

$$\therefore\ V = \frac{mv_0}{M} \qquad \ldots(1)$$

$$L_i = L_f$$

$$\Rightarrow mv_0\frac{L}{2} = \frac{ML^2}{12}\omega \Rightarrow \omega = \frac{m}{M}\frac{6v_0}{L} \qquad \ldots(2)$$

$e = 1$ along line of impact.

$$\therefore\ V + \omega\frac{L}{2} = v_0 \qquad \ldots(3)$$

By (1), (2) and (3), $\frac{m}{M}v_0 + \frac{3m}{M}v_0 = v_0 \ \Rightarrow \frac{4m}{M} = 1$

$$\therefore\ \frac{m}{M} = \frac{1}{4}, \omega = \frac{3v_0}{2L},\ V = \frac{v_0}{4}$$

11.(B) $2(kx)\sin 30° = mg$

$kx = mg$

$kx.\cos 30° = ma_x$

$a_x = \dfrac{\sqrt{3}g}{2}$

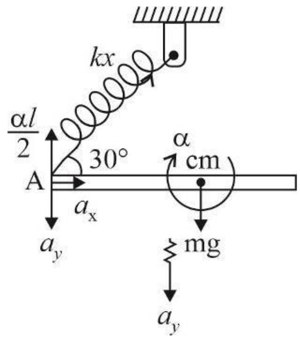
$mg - kx.\sin 30° = ma_y$

$\Rightarrow \; a_y = \dfrac{g}{2}$

$\tau_{cm} = I_{cm}\alpha$

$(kx)\sin 30°, \dfrac{l}{2} = \dfrac{ml^2}{12}\alpha$

$\Rightarrow \quad mg.\dfrac{l}{4} = \dfrac{ml^2}{12}\alpha \qquad \therefore \qquad \alpha = \dfrac{3g}{L}$



12.(A) $a_A = \sqrt{\left(\dfrac{3g}{2} - \dfrac{g}{2}\right)^2 + \left(\dfrac{\sqrt{3}g}{2}\right)^2} = \sqrt{g^2 + \dfrac{3g^2}{4}} = \sqrt{\dfrac{7}{2}}g$

$a_A = 1.3g$

SECTION 2

1.(2) Assume the cavity to be made of +ve and –ve mass. Then it will become a system of a sphere of radius R of the mass (say M) and a sphere of radius $\dfrac{R}{2}$ of –ve mass $\left(\dfrac{-M}{8}\right)$. At centre of cavity, gravitation field is:

$$\vec{E}_1 = \vec{E}_{+\rho} + \vec{E}_{-\rho} = \frac{GM}{R^3}\left(\frac{R}{2}\right) + 0 = \frac{GM}{2R^2}$$

At contact point

$$E_2 = E_{+\rho} + E_{\rho} = \frac{GM}{R^2} + \frac{G(-M/8)}{(R/2)^2} = \frac{GM}{2R^2} \quad \therefore \frac{F_1}{F_2} = \frac{m_1E_1}{m_2E_2} = \frac{m_1}{m_2} = 2$$

2.(11) $L_i = L_f$ about ‘O’

$mv_0a = I_o\omega$

$\Rightarrow \; mv_0a = \left(\dfrac{ma^2}{6} + \dfrac{ma^2}{2}\right)\omega \;\Rightarrow\; mv_0a = \dfrac{4ma^2}{3}\omega \Rightarrow \omega = \dfrac{3v_0}{4a}$

$E_i = E_f \Rightarrow \dfrac{1}{2}I_o\omega^2 + \dfrac{mga}{2} = mg\dfrac{a}{\sqrt{2}} \Rightarrow v_0 = \sqrt{\dfrac{16}{3}\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right)ag} = \sqrt{\dfrac{kag}{10}}$

$\therefore \; \dfrac{k}{10} = 1.1 \Rightarrow k = 11$

2.(4) Applying Bernoulli's theorem at a point just inside and a point just outside the orifice (at same heights)

$$P_a + \frac{1}{2}\rho v_a^2 = P_b + \frac{1}{2}\rho v_b^2$$

$$\Rightarrow \quad P_0 + h\rho_2 g + (h-y)\rho_1 g + \frac{1}{2}\rho_1 (0)^2 = P_0 + \frac{1}{2}\rho_1 v^2 \quad \Rightarrow \quad h(\rho_1 + \rho_2)g - y\rho_1 g = \frac{1}{2}\rho_1 v^2$$

Putting the values we get, $\Rightarrow v = 4m/s$

4.(3.46) mg is non-impulsive force.

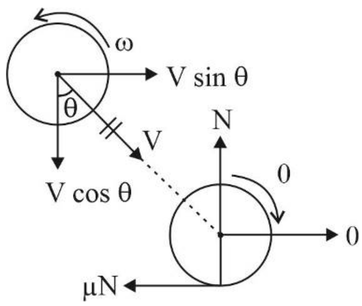
Along vertical direction $\rightarrow$ impulse = change in momentum

$\Rightarrow N.\Delta t = 0 - (-mV\cos\theta)$

$\Rightarrow N\Delta t = mV\cos\theta$ ...(1)

Along horizontal, $\mu N\Delta t = mV\sin\theta$ ...(2)

Now, impulse of torque = change in angular momentum



$\Rightarrow \mu NR\Delta t = \frac{mR^2}{2}.\frac{V}{R}$

$\Rightarrow \mu(N\Delta t) = \frac{mV}{2}$ ...(3)

By (1) and (2), $\mu = \tan\theta$

By (2) and (3), $\sin\theta = \frac{1}{2}$

$\therefore \theta = 30^\circ$ *and* $\mu = \frac{1}{\sqrt{3}}$

$$\frac{\mu}{\theta} = \frac{6}{\pi\sqrt{3}} = \frac{2\sqrt{3}}{\pi} = \frac{3.46}{\pi}$$

5.(40) Let the fraction of Metal A be x

Let the total volume of the sphere be V

Then, Weight = Buoyant force

$$\Rightarrow \quad [8x + 6(1-x)]Vg = \frac{1}{2}(13.6)Vg$$

$$\Rightarrow \quad x = 0.4$$

6.(31.25) $F = \sum \eta A\frac{dv}{dr} = 0.5 \times 0.75\left[\frac{0.5}{10\times10^{-3}} + \frac{0.5}{15\times10^{-3}}\right] = 31.25N$

[CHEMISTRY]

SECTION 1

1.(B) $Ag^+ \quad + \quad 2CN^- \rightleftharpoons Ag(CN)_2^-$

Initial	0.03	0.1	0
Final	x	0.04	0.03

(here assuming nearly complete consumption of Ag^+ due to very high value of K_c)

$$\Rightarrow \quad \frac{0.03}{x(0.04)^2} = 2.5\times10^{18} \quad \Rightarrow \quad x = \frac{300}{16\times2.5\times10^{18}} \quad \Rightarrow \quad x = 7.5\times10^{-18}M$$

2.(C) When $K_p > Q$ rate of forward reaction > rate of backward reaction then forward reaction is spontaneous. When $\Delta G^\circ < RT\ln Q, \Delta G^\circ$ is Positive, reverse reaction is feasible and reaction is non-spontaneous. When $K_p = Q,$ rate of forward reaction = rate of backward reaction hence reaction is in equilibrium. When $T\Delta S > \Delta H, \Delta G$ will be negative only when ΔH is positive. This reaction is spontaneous and endothermic.

3.(D) $C_6H_6(l) + \frac{15}{2}O_2(g) \longrightarrow 6CO_2(g) + 3H_2O(l);$

$C(s) + O_2(g) \longrightarrow CO_2(g); \qquad \Delta_C H = Q_2$

$H_2(g) + \frac{1}{2}O_2(g) \longrightarrow H_2O(l); \qquad \Delta_C H = Q_3$

Required equation: $6C(s) + 3H_2(g) \longrightarrow C_6H_6(l); \Delta H = 6Q_2 + 3Q_3 - Q_1$

4.(A) $\Delta S_m = C_{p,m}\ln\left(\frac{T_2}{T_1}\right) + nR\ln\left(\frac{P_1}{P_2}\right) = \frac{7}{2}R\ln\left(\frac{100}{400}\right) + nR\ln\left(\frac{1}{10}\right)$

$$= -\frac{7}{2}R\times(1.4) - 1\times R\times\ln(5\times2) = -9.8 - 2\times2.3 = -14.4 \text{ cal } K^{-1}mol^{-1}$$

5.(C) Pressure of ammonia = $20\times\frac{60}{100} = 12atm$

$2NH_3(g) \rightleftharpoons N_2(g) + 3H_2(g)$

12 atm 2 atm 6 atm

$$K_p = \frac{2\times(6)^3}{(12)^2} = 3$$

6.(AC) The above reaction is exothermic, so increase in temperature favours the backward direction. Increase in concentrations of NH_3 and CO_2 will move reaction in backward direction.

7.(ABCD)

$\Delta H^\circ = \Delta_f H^\circ(\text{products}) - \Delta_f H^\circ(\text{reactants}) = -166 - (-51) = -115 \text{kJ mol}^{-1}$

$\Delta S^\circ = 266 - 243 = 23 \text{J mol}^{-1} \text{K}^{-1} = 0.023 \text{ kJ mol}^{-1} \text{K}^{-1}$

$\Delta H^\circ = -\text{ve}$

$\Delta S^\circ = +\text{ve}$

$\therefore \quad \Delta G = \Delta H - T\Delta S$

ΔG will be –ve at all temperature and the reaction is favoured at all temperature.

8.(BC) $2C(s) + 3H_2(g) \rightarrow C_2H_6(g)$

$\Delta_f H^0 = \left[2 \times \Delta_{sub} H^0 + 3 \times \text{B.E}(H-H)\right] - \left[\text{B.E.}(C-C) + 6 \times \text{B.E.}(C-H)\right]$

$\Rightarrow -85 = \left[(2 \times 718) + (3 \times 436)\right] - (x + 6y)$

$\therefore \; x + 6y = 2829 \qquad \ldots(1)$

Similarly for $C_3H_8(g)$

$2x + 8y = 4002 \qquad \ldots(2)$

Solving (1) and (2), x = 345 y = 414

9.(AB) $\Delta S = \dfrac{\Delta H}{T} = \dfrac{-(40600) \times 0.5}{373} = -54.42 \text{J} / \text{K}$

10.(AD) $NO(g) + NO_2(g) \underset{}{\overset{K_{P_1}}{\rightleftharpoons}} N_2O_3(g)$

	$NO(g)$	$NO_2(g)$	$N_2O_3(g)$
Initially	3P	5P	0
At equilibrium	3P – x	5P – x – 2y	x

$2NO_2(g) \rightleftharpoons N_2O_4(g)$

$5P - x - 2y \qquad\qquad y$

$K_{P_2} = \dfrac{P_{N_2O_4}}{(P_{NO_2})^2}, \; 8 = \dfrac{P_{N_2O_4}}{(0.5)^2}$

$P_{N_2O_4} = 2 \text{atm} \quad \therefore \quad y = 2 \text{atm}$

$P_{total} = P_{NO} + P_{NO_2} + P_{N_2O_3} + P_{N_2O_4}$

$5.5 = 3P - x + 0.5 + x + 2$

$3P = 3 \quad \Rightarrow \quad P = 1$

$5P - x - 2y = 0.5$

$5 \times 1 - x - 2 \times 2 = 0.5, \; x = 0.5 \text{atm}$

$K_{P_1} = \dfrac{P_{N_2O_3}}{(P_{NO})(P_{NO_2})} = \dfrac{0.5}{2.5 \times 0.5} = 0.4 \text{ atm}^{-1}$

$P_{NO} = 2.5 \text{atm}, \; P_{NO_2} = 0.5 \text{atm}, \; P_{N_2O_3} = 0.5 \text{atm}, \; P_{N_2O_4} = 2 \text{atm}$

11.(B) $\Delta_r H = -1575 + \frac{3}{2}[-241.8] - [-2021]$

$= -1575 - 362.7 + 2021$

$= +83.3\,kJ$ [per mole of gypsum]

As $\Delta_r H$ is 83.3 kJ per 172 g of gypsum thus per kg of gypsum that will be $\frac{83.3}{0.172}kJ = 484.3\,kJ$

12.(A) $p_{H_2O} = 1$

Hence $K_p = (p_{H_2O})^{1.5} = 1$

As $\Delta G^\circ = -RT\,\ell n\,K \quad \Rightarrow \quad \Delta G^\circ = 0$

$\Delta H^\circ = T\Delta S^\circ \quad \Rightarrow \quad T = \frac{\Delta H^\circ}{\Delta S^\circ} = \frac{83300}{219.4} \approx 380K = 107°C$

SECTION 2

1.(354) $2.303\log\frac{P_2}{P_1} = \frac{\Delta H}{R}\left(\frac{T_2 - T_1}{T_1 T_2}\right)$

$\underset{(373K)}{T_2 = 100°C}; \quad P_2 = 760mm\,(1atm)$

$T_1 = ? \qquad P_1 = 380\,mm$

$\Delta H = 540 \times 18\,cal/mol$

$2.303\log\frac{760}{380} = \frac{540 \times 18}{2}\left(\frac{373 - T_1}{T_1 \times 373}\right)$

$\Rightarrow \quad 2.303\log 2 = 540 \times 9\left(\frac{373 - T_1}{T_1 \times 373}\right) \quad \Rightarrow \quad 0.693 = 4860\left(\frac{373 - T_1}{373T_1}\right)$

$\Rightarrow \quad \frac{0.693 \times 373T_1}{4860} = 373 - T_1 \quad \Rightarrow \quad T_1 = 354.226\,K \approx 354\,K$

2.(6.92)

$NH_4HS(s) \rightleftharpoons NH_3(g) + H_2S(g)$

$2 - x \qquad x - y \qquad x$

$NH_3(g) \rightleftharpoons \frac{1}{2}N_2(g) + \frac{3}{2}H_2(g)$

$x - y \qquad y/2 \qquad 3y/2$

Given,

$2 - x = 1 \qquad \frac{3y}{2} = 0.75$

$x = 1 \qquad y = \left(\frac{1}{2}\right)$

$$K_{C_1} = x = \frac{0.5}{2} \times \frac{1}{2} = \frac{1}{8} M^2$$

$$y = K_{C_2} = \frac{\left(\frac{3}{8}\right)^{3/2}\left(\frac{1}{8}\right)^{1/2}}{\left(\frac{1}{4}\right)} = \left(\frac{3\sqrt{3}}{16}\right)$$

$$a = [NH_3] = \frac{1}{4} M$$

$$[H_2S] = \frac{1}{2} M$$

$$b = [N_2] = \frac{1}{8} M$$

$$[H_2] = \frac{3}{8} M$$

$$\frac{y}{x(a+b)} = \frac{3\sqrt{3}}{16} \times \frac{8}{1} \times \frac{8}{3} = 4\sqrt{3} = 4 \times 1.73 = 6.92$$

3.(2)

	$A \rightleftharpoons$	B	$+ \; C$
Initial	10	0	0
Final	$(10-x)$	x	x

$$M^{\circ}_{avg} = \frac{(10-x)100 + x(60) + x(40)}{(10+x)} = 83$$

$$= (10+x)83 = (10-x)100 + 100x = 10(100) - 100x + 100x$$

$$\Rightarrow \quad 830 + 83x = 1000 \quad \Rightarrow \quad x = \frac{1000-830}{83} = \frac{170}{83} \approx 2$$

4.(29)

	$S_2Cl_4(g) \rightleftharpoons$	$2SCl_2(g)$
Initial	4	0
At equilibrium	0.2	7.6

$$\therefore \; [S_2Cl_4] = \frac{0.2}{10} \qquad [SCl_2] = \frac{7.6}{10}$$

$$K_c = \frac{[SCl_2]^2}{[S_2Cl_4]} = \frac{\frac{7.6}{10} \times \frac{7.6}{10}}{\frac{0.2}{10}} = 28.88 = 29.$$

5.(4)

	$2HI(g) \longrightarrow$	$H_2(g)$	$+ \; I_2(g)$
Initially	4	0	0
At equilibrium	$4-x$	$\frac{x}{2}$	$\frac{x}{2}$

$$K' = \frac{\frac{1}{2}.\frac{1}{2}}{(4-1)^2} = \frac{1}{36} \qquad \left(\therefore \frac{x}{2} = 0.5\right)$$

$$H_2 + I_2 \rightleftharpoons 2HI$$

$$K_p = \frac{1}{K'} = \frac{1}{\frac{1}{36}} = 36; \; y = 4$$

6.(8) $H_2(g) + Cl_2(g) \rightarrow 2HCl(g) \qquad \Delta G^\circ = -191\,kJ/mol$

$$\Delta_r G^\circ = \Delta_r H^\circ - T\Delta_r S^\circ$$

$$\Delta_r H^0 = 2\Delta_f H^0_{HCl} - \left[\Delta_f H^0_{H_2(g)} + \Delta_f H^0_{Cl_2(g)}\right]$$

$$\Delta_r H^0 = -92.5kJ \times 2$$

$$\Delta_r H^0_{rxn} = -185kJ$$

To find $\Delta_r S^\circ$

$$\Delta_r G^\circ = \Delta_r H^\circ - T\Delta_r S^\circ$$

$$-191 = -185 - 300\Delta_r S^\circ$$

$$\Delta_r S^\circ = 20JK^{-1}mol^{-1}$$

$$\Delta_r S^\circ = 2S^0_{HCl} - \left[S^0_{H_2} + S^0_{Cl_2}\right]$$

$$20JK^{-1}mol^{-1} = 2 \times 184JK^{-1}mol^{-1} - \left[S^0_{H_2} + 228JK^{-1}mol^{-1}\right]$$

$$S^0_{H_2} = +120JK^{-1}mol^{-1}$$

$$\frac{S^0_{H_2}}{15} = \frac{120}{15} = 8$$

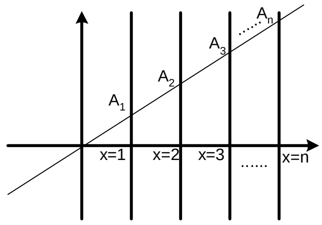
[MATHEMATICS]

SECTION 1

1.(B) Equation of the line in parametric form is $\dfrac{x}{\frac{1}{\sqrt{6}}}=\dfrac{y}{\frac{\sqrt{5}}{\sqrt{6}}}=r.$

Any point on the given line can be taken as $\left(\dfrac{r}{\sqrt{6}},\dfrac{\sqrt{5}}{\sqrt{6}}r\right).$

Now, if r = OA then the point $\left(\dfrac{r}{\sqrt{6}},\dfrac{\sqrt{5}}{\sqrt{6}}r\right)$ will also satisfy the equation $x = 1$.



$\Rightarrow\ r=\sqrt{6}=OA_1$

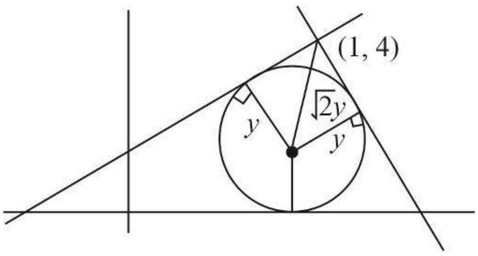
Similarly $OA_2=2\sqrt{6}.$ Hence

$OA_1^2+OA_2^2+....+OA_3^2+.....+OA_n^2$

$=6.1^2+2^26+3^26+......+n^26$

$=6\left(\dfrac{n(n+1)(2n+1)}{6}\right)=2n^3+3n^2+n$

2.(A) $(x-1)^2+(y-4)^2=\left(y\sqrt{2}\right)^2$



$x^2-2x+1+y^2-8y+16=2y^2$

3.(A) $\because\ d(P,BC)\le\min\{d(P,AB),d(P,AC)\}$

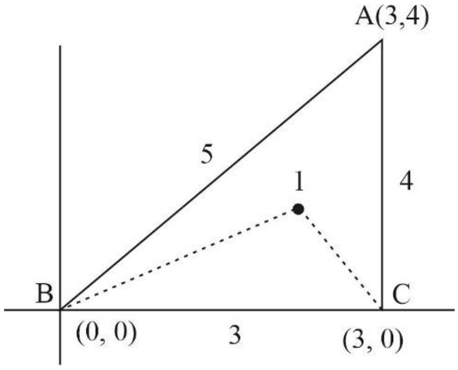
$\Rightarrow\ d(P,BC)\le d(P,AB)$ *and* $d(P,BC)\le d(P,AC)$

$\Rightarrow\ d(P,BC)$ is maximum when

$\Rightarrow\ d(P,BC)=d(P,AB)=d(P,AC)$

$\Rightarrow$ P is the incentre of ΔABC

$\Rightarrow$ Maximum value of $d(P,BC)$



$\Rightarrow$ y-coordinate of the incentre of $\Delta ABC = \dfrac{3\times4+4\times0+5\times0}{3+4+5}=1$

4.(A) Combined equation of pair of lines through the origin joining the points of intersection of line $y=\sqrt{k}x+1$ with the given curve is

$$x^2+2xy+\left(2+\sin^2\beta\right)y^2-\left(y-\sqrt{k}x\right)^2=0$$

For the chord to subtend a right angle at the origin $(1-k)+\left(2+\sin^2\beta-1\right)=0$

[As sum of the coefficients of $x^2 and\ y^2=0$]

$\Rightarrow \quad \sin^2\beta=k-2 \quad \Rightarrow \quad 0\le k-2\le1 \quad \Rightarrow \quad 2\le k\le3$

5.(A) Let point P on the straight line x + y = 4 be (m, 4 – m), this will be nearest to the parabola if $\perp$ at this point to the straight line becomes normal to the parabola.

Let it is normal at $x-10=t^2, y=2t$

Perpendicular to $x+y=4\ at\ (m, 4-m)$ is $y-(4-m)=(x-m)$ …(1)

Normal at parabola at $\left(t^2+10, 2t\right)$ is $y+t(x-10)=12t+t^3$ …(2)

(1) and (2) are same $\Rightarrow$ $t=-1, m=\dfrac{17}{2}$ so required point is $\left(\dfrac{17}{2},-\dfrac{9}{2}\right)$

6.(AB) Solution. Focus is $\left(\dfrac{7}{2},\dfrac{7}{2}\right)$ and its axis is the line $y = x$. Corresponding value of 'a' $=\dfrac{1}{4}\left(\sqrt{1+1}\right)=\dfrac{\sqrt{2}}{4}$.

Let the equation of its directrix be $y+x+\lambda=0$.

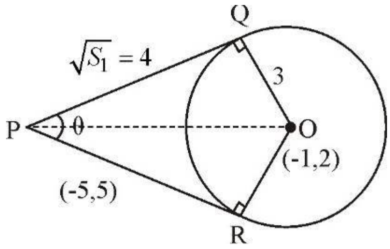
$\Rightarrow \quad \dfrac{|3+4+\lambda|}{\sqrt{2}}=2.\dfrac{\sqrt{2}}{4}$

$\Rightarrow \quad \lambda=-6,-8$

Thus, equation of parabola is $\left(x-\dfrac{7}{2}\right)^2+\left(y-\dfrac{7}{2}\right)^2=\dfrac{(x+y-6)^2}{2}$

or $\quad \left(x-\dfrac{7}{2}\right)^2+\left(y-\dfrac{7}{2}\right)^2=\dfrac{(x+y-8)^2}{2}$

7.(ACD) Angle between PQ and PR $\Rightarrow \theta = 2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$



$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{a}{\sqrt{S_1}} \quad \Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{3}{4}$

$\therefore \sin\left(\frac{\theta}{2}\right) = \frac{3}{5}$; $\cos\left(\frac{\theta}{2}\right) = \frac{4}{5}$

$\therefore \sin\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

$\theta = \sin^{-1}\left(\frac{24}{25}\right)$

Area of $\Delta PQR = \frac{1}{2}\times PQ\times PR\times\sin\theta = \frac{1}{2}\times4\times4\times\frac{24}{25} = \frac{192}{25}$

Area of quadrilateral $PAOB = 2\times\left(\frac{1}{2}\times PQ\times QO\right) \quad = 2\times\left(\frac{1}{2}\times3\times4\right) = 12$ sq. unit

Angle between QR and $QP = \alpha = 90° - \frac{\theta}{2}$

$\therefore \sin\alpha = \sin\left(90° - \frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) = \frac{4}{5} \Rightarrow \alpha = \sin^{-1}\left(\frac{4}{5}\right)$

8.(ABC) Circle on PQ as diameter

$\left(x - t_1^2\right)\left(x - t_2^2\right) + \left(y - 2t_1\right)\left(y - 2t_2\right) = 0$

$x^2 + y^2 - \left(t_1^2 + t_2^2\right)x - 2\left(t_1 + t_2\right)y + t_1^2t_2^2 + 4t_1t_2 = 0$

Put $\frac{t_1 + t_2}{2} = 1$ *and* $t_1t_2 = -1$

Hence equation of circle become

$x^2 + y^2 - 6x - 4y - 3 = 0$ …(1)

With centre (3, 2) and radius 4

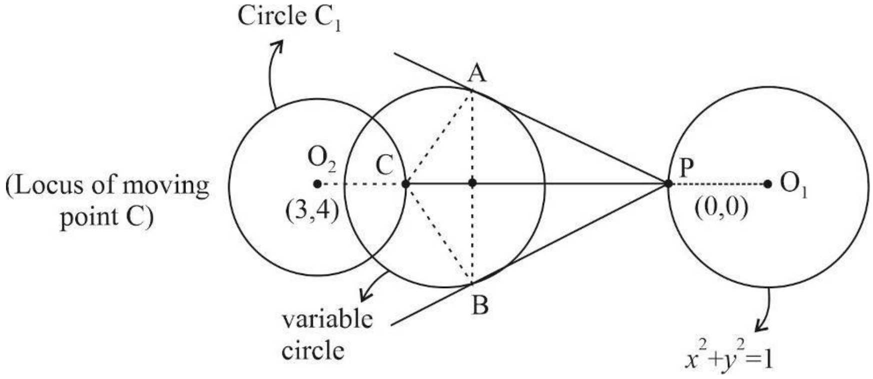
Equation (1) is not orthogonal to $x^2 + y^2 + 2x - 6y + 3 = 0$

9.(ABCD) Quadrilateral PACB is cyclic and PC will be the diameter of any circle passing through any of given 4 points.

$\therefore$ Diameter will be PC.

Locus of C is $(x-3)^2 + (y-4)^2 = 1 \equiv C_1$

$\therefore$ PC is the distance between two circles.



$\therefore$ Maximum distance will be $O_1O_2 + r_1 + r_2 = 7.$

Minimum distance will be $O_1O_2 - r_1 - r_2 = 3.$

Length of direct common tangent = $\sqrt{(O_1O_2)^2 - (r_1 - r_2)^2} = 5$

10.(AC) $\frac{2}{3}x^2 + \frac{p}{3}xy + y^2 = (y - mx)(y - m'x)$

and $\frac{2}{-3}x^2 + \frac{q}{-3}xy + y^2 = \left(y + \frac{1}{m}x\right)(y - m'x)$

then, $m + m' = \frac{-p}{3}$, $mm' = \frac{2}{3}$

$\frac{1}{m} - m' = \frac{-q}{3}$, $\frac{m'}{m} = \frac{2}{3}$

$m^2 = 1 \Rightarrow m = \pm 1$

If $m = 1, m' = \frac{2}{3}$ so $p = -5, q = -1$; If $m = -1, m' = -\frac{2}{3}$ so $p = 5, q = 1$

11.(B) & 12.(C)

$t_1 = \sqrt{h^2 + k^2 - 4}$; $t_1^2 = h^2 + k^2 - 4$; $t_2 = \sqrt{h^2 + k^2 - 4h}$; $t_2^2 = h^2 + k^2 - 4h$; $t_3 = \sqrt{h^2 + k^2 - 4k}$; $t_3^2 = h^2 + k^2 - 4k$

Further, $t_1^4 = t_2^2 \cdot t_3^2 + 16$

$\Rightarrow (h^2 + k^2 - 4)^2 = (h^2 + k^2 - 4h)(h^2 + k^2 - 4k) + 16$

$\Rightarrow (h^2 + k^2)^2 + 16 - 8(h^2 + k^2) = (h^2 + k^2) - 4h(h^2 + k^2)^2 - 4k(h^2 + k^2) + 16hk + 16$

$\Rightarrow (h^2 + k^2)(4h + 4k) - 8(h^2 + k^2 + 2hk) = 0$

$\Rightarrow (h^2 + k^2)(h + k) - 2(h + k)^2 = 0 \Rightarrow (h + k)(h^2 + k^2 - 2(h + k)) = 0$

So, $L_1 \Rightarrow x + y = 0$; slope of $L_1 = -1$

$C_1 \Rightarrow x^2 + y^2 - 2x - 2y = 0.$ centre $\equiv (1,1)$ radius $= \sqrt{2}$

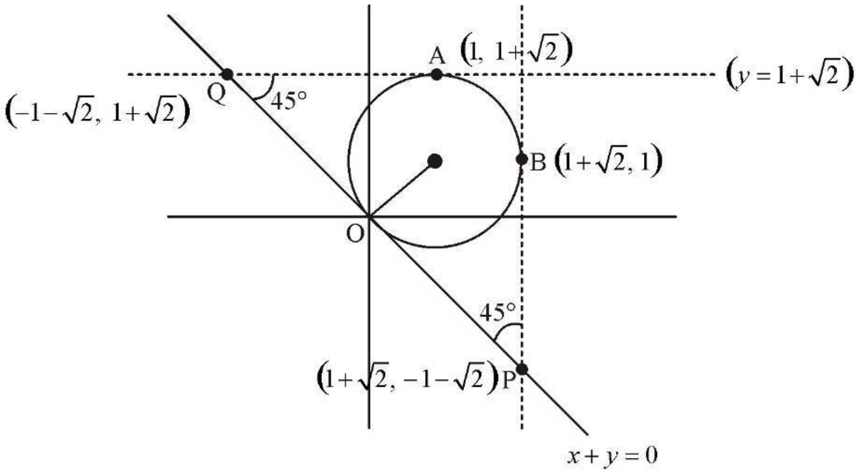
$L_1 \& C_1$ passes through origin

As line are at an angle of 45° with L_1

So, slope of those lines be m_1 & m_2

$m_1 = \frac{-1 + \tan 45°}{1 - (-1)\tan 45°} = 0 \Rightarrow$ line is parallel to x-axis

$m_2 = \frac{-1 - \tan 45°}{1 + (-1)\tan 45°} \Rightarrow$ line is parallel to y-axis



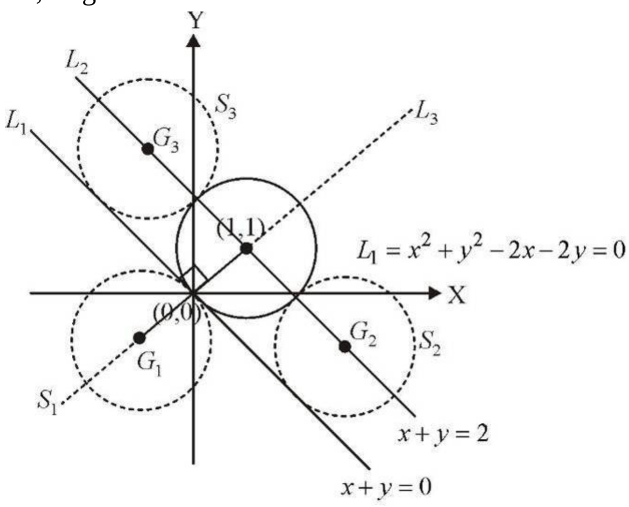
As lines are perpendicular to each other, so circum centre of a right angled triangle lies on mid-point of hypotenuse

$P(1+\sqrt{2}, -1-\sqrt{2})$ and $Q(-1-\sqrt{2}, 1+\sqrt{2})$ are ends of hypotenuse

So, circumcentre would be (x_1, y_1)

$x_1 = \frac{1+\sqrt{2}+(-1-\sqrt{2})}{2} = 0$; $y_1 = \frac{(-1-\sqrt{2})+1+\sqrt{2}}{2} = 0$

So, origin is the circumcentre



As S_1, S_2 & S_3 are circles congruent to C_1 & touch both L_1 & C_1, so there are only three options

So, centre of circles S_3, C_1 & S_2 will be collinear lying on line L_2

As L_2 is parallel to L_1

So, $L_2 \equiv x+y+k=0$

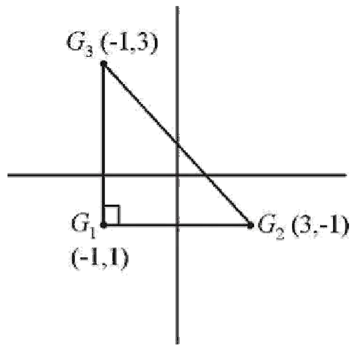
Satisfying (1,1) on L_2, we get $(x+y)=2$

Also, centre of S_1, C_1 and origin will be collinear lying on line $L_3 \equiv y-x=0$

So, centre of S_3 & S_2 could be found out by using parametric form.

As centre of S_3 & S_2 are at a distance of 2 radius i.e., $2\sqrt{2}$ from (1,1)

So, $G_3 = (1+2\sqrt{2}\times\cos135°, 1+2\sqrt{2}\times\sin135°) = (-1,3)$



$G_2 = \left(1 - 2\sqrt{2} \times \cos 135°, 1 - 2\sqrt{2} \times \sin 135°\right) = (3, -1)$

Also, $G_1 = \left(0 - \sqrt{2}\cos 45°, 0 - \sqrt{2}\sin 45°\right) = (-1, -1)$

Area of $\Delta\left(G_1 G_2 G_3\right) = \frac{1}{2} \times 4 \times 4 = 8$

SECTION 2

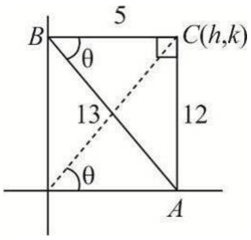
1.(73) $x^2 + y^2 - 14x - 6y - 6 = 0$

General point $\left(7 + 8\cos\theta,\ 3 + 8\sin\theta\right)$

$3x + 4y = 3\left(7 + 8\cos\theta\right) + 4\left(3 + 8\sin\theta\right)$

$\Rightarrow\ 33 + 24\cos\theta + 32\sin\theta \Rightarrow 33 + 8\left(3\cos\theta + 4\sin\theta\right) \Rightarrow$ maximum value = 73

2.(5)



$\tan\theta = \frac{k}{h} = \frac{12}{5}$; $12x - 5y = 0 \Rightarrow \quad k = 5$

3.(2) Equation of 2 circles passing through A(0, *a*) and B(0, –*a*) is $S_0 + \lambda L = 0$

$\Rightarrow \quad (x^2 + y^2 - a^2) + \lambda x = 0$

Touches line $y = mx + c$

Apply $p = r$

$$\left|\frac{m\left(\frac{-\lambda}{2}\right) + (-1)\cdot 0 + c}{\sqrt{m^2+1}}\right| = \sqrt{\frac{\lambda^2}{4} + a^2}$$

$\lambda^2 + 4\lambda cm + 4a^2m^2 + 4a^2 - 4c^2 = 0 \qquad$ ………..(i)

Condition of orthogonally $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

For two circles:

$x^2 + y^2 + \lambda_1 x - a^2 = 0$ and $x^2 + y^2 + \lambda_2 x - a^2 = 0$

$2 \cdot \frac{\lambda_1}{2} . \frac{\lambda_2}{2} + 2 \cdot 0 \cdot 0 = -a^2 - a^2$

$\lambda_1\lambda_2 = -4a^2$

From (i)

$\lambda_1\lambda_2 = 4a^2m^2 + 4a^2 - 4c^2$

$\Rightarrow \quad 4a^2m^2 + 4a^2 - 4c^2 = -4a^2 \Rightarrow a^2(m^2 + 2) = c^2 \qquad \therefore\ k = 2$

4.(9) Since the given circle is $(x+5)^2+(y-12)^2=(14)^2$ or $\left(\frac{x+5}{14}\right)^2+\left(\frac{y-12}{14}\right)^2=1$

Let $\frac{x+5}{14}=\cos\theta$ and $\frac{y-12}{14}=\sin\theta$

$\Rightarrow \quad x=14\cos\theta-5$ and $y=14\sin\theta+12$

$x^2+y^2=(14\cos\theta-5)^2+(14\sin\theta+12)^2$

$=196\cos^2\theta-140\cos\theta+25+196\sin^2\theta+336\sin\theta+144$

$=196+25+144+336\sin\theta-140\cos\theta=365+28(12\sin\theta-5\cos\theta)$

Maximum value of $x^2+y^2=365+28\times 13=729$

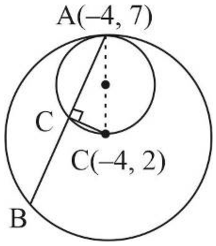
$\because \quad -\sqrt{12^2+5^2}\le 12\sin\theta-5\cos\theta\le +\sqrt{12^2+5^2}$

Maximum value of $(x^2+y^2)^{1/3}=(729)^{1/3}=9^{3\times\frac{1}{3}}=9$

5.(4) Let O be (–4, 2)

OC is perpendicular to AC as OA is diameter

$\Rightarrow$ C bisects chord AB



As O is centre of bigger circle

$\Rightarrow$ AC = 4 units

6.(576) The line $y=\sqrt{3}x$ is passing through the origin and slope is $\sqrt{3}$, hence in parametric form the equation of given line can be written as

$$\frac{x}{1/2}=\frac{y}{\sqrt{3}/2}=r \qquad \text{............(1)}$$

Any point on the line (1) is $\left(\frac{r}{2},\frac{\sqrt{3}r}{2}\right)$. If the line cuts the given curve, then

$\frac{r^3}{8}+\frac{ar^2}{4}+\frac{br}{2}-72=\frac{\sqrt{3}r}{2}$. This is a cubic equation in *r*. Roots of this equation r_1, r_2, r_3 will represent OA, OB and OC.

Therefore $OA.OB.OC=|r_1||r_2||r_3|=576$